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418. Proposed by B. F. FINKEL, Drury College.

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs \$200 an acre, what will be the least possible cost of the land required?

Granville's *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

SOLUTION BY CLARIBEL KENDALL, University of Colorado.

For the cost of the land in this problem to be a minimum the area must be a minimum.

Let x = radius of ends in yards, y = length of straightaway sides in yards, and v = value of land.

Then $(2x + y)2x$ = area of the rectangular tract, and $70y$ = area of grandstands.

(1) $A = 70y + (2x + y)2x$, total area to be bought.

(2) $2\pi x + 2y = 440$ yds., length of the track, or $y = 220 - \pi x$, $y' = -\pi$, where $y' = dy/dx$.

(3) $A' = 70y' + 8x + 2xy' + 2y = 0$; or, substituting for y and y',

$$(4 - 2\pi)x = 35\pi - 220, \quad \text{and} \quad x = \frac{220 - 35\pi}{2\pi - 4} = \frac{385}{8} \text{ yds.} \left(\pi = \frac{22}{7}\right).$$

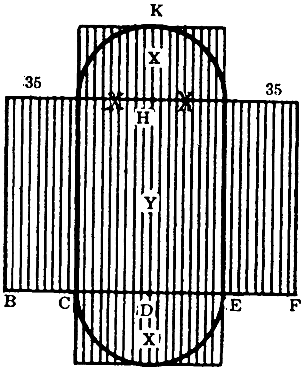

FIG. 1.

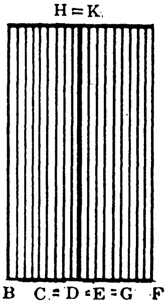

FIG. 2.

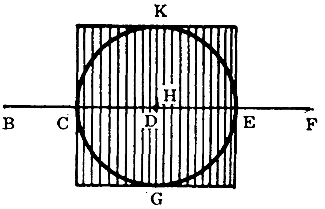

FIG. 3.

Whence

$$y = \frac{275}{4} \text{ yds.,} \quad \text{and} \quad A = 70y + 2x(2x + y) = \frac{824775}{4} \text{ sq. yds.} = 4.28 \text{ acres.}$$

At \$200 an acre, the cost v is \$856 as given by Granville.

From (3) above

$$A'' = 8 - 4\pi,$$

showing that A'' is a negative constant. This proves that A is a maximum, and that no minimum is even possible, under the customary definitions of maxima and minima.

We next inquire whether there may not be, after all, some sort of practical minimum, based on the exclusion of negative values for the end-radius x and for the straightaway side y.

From (2) above $\pi x + y = 220$. Plainly, x must lie between 0 and 70, while y must lie between 220 and 0.

When $x = 0$, $y = 220$; $A = 15400$, $v = \$636.36$. The land bought is now a rectangle. The semi-circular ends have degenerated into points and the track is a strip of zero width between the front edges of the grandstands.

When $x = 70$, $y = 0$; $A = 19600$, $v = \$809.92$, and the ground bought is now a square, the track is the inscribed circle, while the stands reduce to the 35-foot extensions of one of the two diameters of contact.

As x increases from 0 to 70, A and v increase up to the maximum and then decrease. The values for A and v are greater, however, for $x = 70$ than for $x = 0$. (See Fig. 4.) A practical minimum for v may therefore be secured by taking x as small as possible to allow for an easy-running curve at the ends.

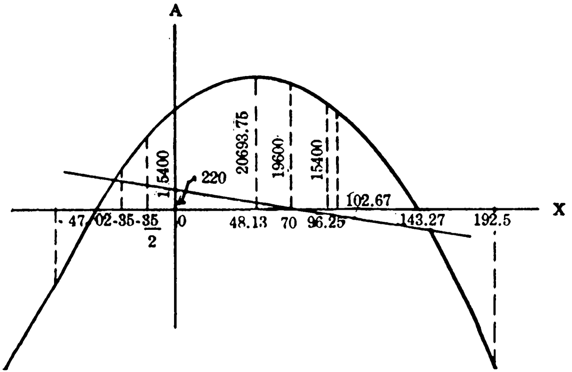

FIG. 4.

As far as has been discovered, there is no contradiction involved in the solution, but rather a misconception in the statement of the problem.

Aside from the question of determining a maximum or a minimum, this problem is curious when we try to represent the area for negative values of x and y. We will trace x from $-\infty$ to $+\infty$.

The following conventions will be adopted: Positive areas will be designated by vertical shadings; negative areas by horizontal shadings, positive areas counted twice by diagonal cross-bar shadings; positive and negative areas cancelling one another by vertical and horizontal shadings (negative areas counted twice do not occur). The positive part of the race track will be designated by a heavy solid line, the negative part by a heavy dotted line. In the following figures, as in Figs. 1, 2, 3, $BC(=EF=35)$ the width of the grandstands will be considered as essentially positive and these grandstands always face in opposite directions; $x=CD=DE$, with the usual conventions as to directed line segments, positive when measured to the right and negative when measured to the left; $y=DH$, positive when measured upward, negative when measured downward; $70+2x=BF$, $2x+y=GK$, with the same sign-conventions as above. One of the semi-circular ends of the track always goes from C to E and correspondingly for the other end.

The area of the tract of land may be represented in two ways:

$$A=(70+2x)y+4x^2, \tag{1a}$$

$$A=70y+(2x+y)2x. \tag{1b}$$

$4x^2$ represents the area of the two rectangles, including the semi-circular ends of the race-track; $(70+2x)y$ is the area of the rectangle having BF and DH as dimensions; $70y$ is the area of the two grandstands; $(2x+y)2x$ is the area of the rectangle having CE and GK as dimensions.

For $-\infty<x<0$, $y>0$ it is preferable to consider A in the form (1a). $4x^2$ is always positive and since $y>0$,

$$(70+2x)y \gtreqless 0 \text{ according as } 70+2x \gtreqless 0,\ i.e., \text{ for } x \gtreqless -35. \tag{4}$$

For $0 \leqq x \leqq 70$, $y \geqq 0$ either form is applicable since all the above areas are then positive or zero.

For $70<x<+\infty$, $y<0$ the form (1b) is preferable. $70y$ is always negative, and since $x>0$,

$$(2x+y)2x \gtreqless 0 \text{ according as } 2x+y \gtreqless 0,\ i.e., \text{ for } y \gtreqless -2x(=-385) \text{ or } x \gtreqless \tfrac{385}{2}(=192.5), \tag{5}$$

obtained by substituting for y in equation (2).

In the following constant reference should be made to the graph in Fig. 4.

I. When $x<-47.02$ then $y>0$ and $A<0$. The tract of land is represented by Fig. 5, $(70+2x)y<0$ by (4) since $x<-35$. And since $A<0$, $|(70+2x)y|>4x^2$, *i. e.*, the area of the rectangle $BF \times DH$ is greater than the area of the rectangles including the semi-circular ends of the race track.

II. When $x = -47.02$ then $y > 0$ and $A = 0$. Fig. 5 is again representative, since $x < -35$ but now $|(70 + 2x)y| = 4x^2$, *i. e.*, the positive and negative areas just balance.

III. When $-47.02 < x < 0$ then $y > 0$ and $A > 0$. This must be divided into three special cases dependent upon the three parts of (4).

(*a*) When $-47.02 < x < -35$, $(70 + 2x)y < 0$ by (4) but now $|(70 + 2x)y| < 4x^2$ since $A > 0$, *i. e.*, the positive part of the area predominates. Fig. 5 is again representative.

(*b*) When $x = -35$, $(70 + 2x)y = 0$ by (4) and $A = 4x^2$, a positive quantity. Fig. 6 represents this condition.

(*c*) When $-35 < x < 0$, $(70 + 2x)y > 0$ by (4). All of the rectangular areas are now positive. Figures 7, 8, 9 represent the tract for different values of x, as shown under the cuts.

In Fig. 8, the grandstands occupy the same base but face in opposite directions.

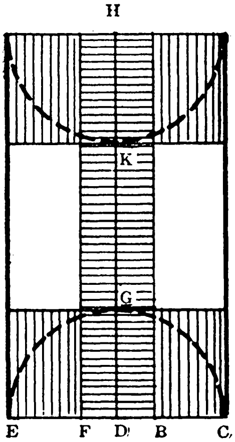

FIG. 5.
$-\infty < x < -35$.

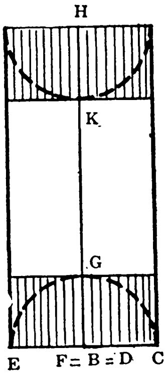

FIG. 6.
$x = -35$.

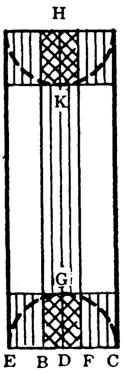

FIG. 7.
$-35 < x < -\frac{35}{2}$.

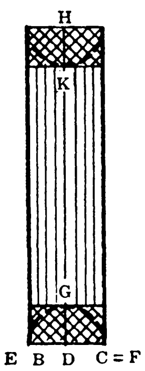

FIG. 8.
$x = -\frac{35}{2}$.

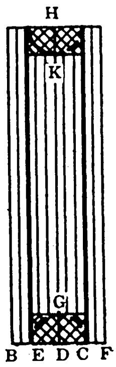

FIG. 9.
$-\frac{35}{2} < x < 0$.

In I, II, III the race-track is partly positive and partly negative. In equation (2), $440 = 2y + 2\pi x$, the $2y$ is positive and the $2\pi x$ is negative, as indicated in the figures by the solid and dotted parts of the track. Whether the negative part might mean that the runner is running backwards has not been investigated.

IV. When $0 \leqq x \leqq 70$ then $y \geqq 0$ and $A > 0$. These cases have been discussed in the solution of the original problem. As stated above either form of (1) may be used.

In IV the entire race-track is positive.

V. When $70 < x < 143.27$ then $y < 0$ and $A > 0$. Since $x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A > 0$, $(2x + y)2x > |70y|$, *i. e.*, the area of the rectangle $GK \times CE$ is greater than the area of the grandstands. This divides into several cases for

(6) $y \gtreqless -x$, *i. e.*, $x \lesseqgtr \frac{308}{3} (= 102.67)$ (obtained by substituting for y in (2))

since $x = 102.67$ lies in the interval under discussion. Figures 10, 11, 12 are representative for different values of x, as shown under the cuts.

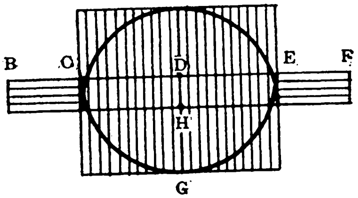

FIG. 10.
$70 < x < 102.67$, $y > -x$.

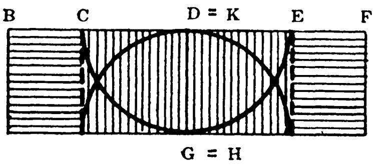

FIG. 11.
$x = 102.67$, $y = -x$.

VI. When $x = 143.27$ then $y < 0$ and $A = 0$. Since $x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A = 0$, $(2x + y)2x = |70y|$. Fig. 12 is again representative and the positive and negative areas just balance.

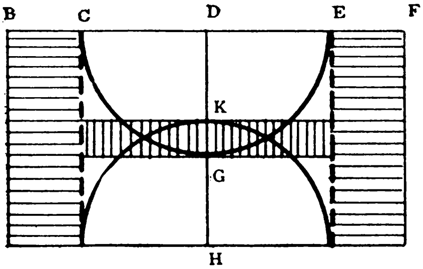

Fig. 12.
$102.67 < x < 192.5, y < -x$.

VII. When $x > 143.27$ then $y < 0$ and $A < 0$. On account of (5) we must consider three cases.

(*a*) When $143.27 < x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A < 0$, $(2x + y)2x < |70y|$, *i. e.*, the area of the rectangle $GK \times CE$ is less than the area of the grandstands. Fig. 12 is again representative.

(*b*) When $x = 192.5$, $(2x + y)2x = 0$ by (5) and $A = 70y$, a negative area. Fig. 13 represents the tract.

(*c*) When $x > 192.5$, $(2x + y)2x < 0$ by (5). Now all of the areas included in A are negative and Fig. 14 is representative.

In V, VI, VII, the race-track is partly positive and partly negative. In equation (2) the $2\pi x$ is positive and the $2y$ is negative as is indicated in the figures by the solid and dotted parts of the track.

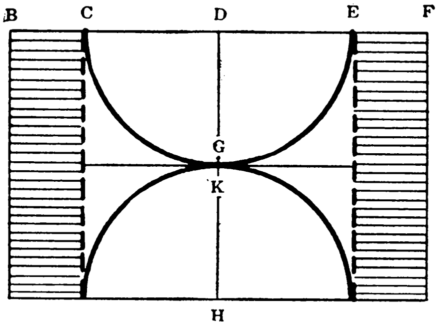

Fig. 13.
$x = 192.5, y = -2x$.

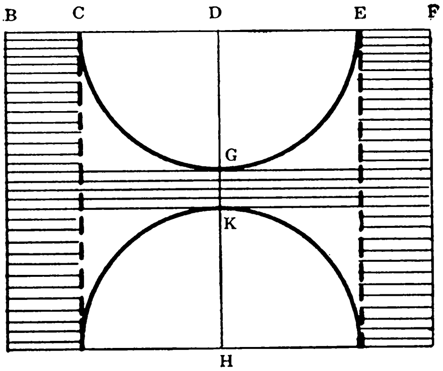

Fig. 14.
$192.5 < x + \infty, y < -2x$.

Also solved by Horace Olson, S. E. Urner, Paul Capron, and J. W. Baldwin.

MECHANICS.

336. Proposed by C. N. SCHMALL, New York City.

An inclined plane, length l, makes an angle $\phi < (\frac{1}{4}\pi)$ with the horizontal plane through its foot. From its foot, a body is projected upward along the plane, with a velocity equal to that of a falling body at the height h, so as to pass over the top and strike the horizontal plane at the maximum distance, x, from the foot of the inclined plane. Show by methods of the calculus that $x = h/(\sin\phi\cos\phi)$, and that the corresponding value of l is $(2h\cot 2\phi)/\cos\phi$.

Solution by the Proposer.

Let v_1 = velocity at time of projection; v = velocity on reaching top of plane; a = height of inclined plane; x_1 = horizontal distance traversed by body after leaving plane.

Then, by the given conditions, we have

$$v_1^2 = 2gh, \tag{1}$$